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# Evaluating the Wald Entropy from two-derivative terms in quadratic actions

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**Abstract**

We evaluate the Wald Noether charge entropy for a black hole in generalized theories of gravity. Expanding the Lagrangian to second order in gravitational perturbations, we show that contributions to the entropy density originate only from the coefficients of two-derivative terms. The same considerations are extended to include matter fields and to show that arbitrary powers of matter fields and their symmetrized covariant derivatives cannot contribute to the entropy density. We also explain how to use the linearized gravitational field equation rather than quadratic actions to obtain the same results. Several explicit examples are presented that allow us to clarify subtle points in the derivation and application of our method.

# 1 Introduction

The Bekenstein–Hawking law [1, 2], relates the entropy $S_{BH}$ to the horizon area $A$ in units of Newton's constant for a black hole in Einstein's theory of gravity,

$$S_{BH} \;=\; \frac{A}{4G_N} \;\; . \tag{1}$$

This relation suggests that the entropy $S$ should be purely geometric, defined strictly at the black hole horizon and should satisfy the first law of black hole mechanics,

$$T_H dS \;=\; dM \; . \tag{2}$$

Here, $M$ is the conserved or ADM mass of the black hole (other conserved charges have been neglected for simplicity) and $T_H = \kappa/2\pi$ is the Hawking temperature in terms of the surface gravity $\kappa$.

The mass $M$ and $\kappa$ are well defined for a stationary black hole in any theory of gravity, and so their definitions do not need to be modified. Wald [3, 4] proposed a definition of the entropy that fulfills all of the above requirements for general theories of gravity. The Wald entropy $S_W$ has a clear geometric interpretation through its identification with the Noether charge for spacetime diffeomorphisms. Further, $S_W$ can always be cast as a closed integral over a cross-section of the horizon $\mathcal{H}$,

$$S_W = \oint_{\mathcal{H}} s_W dA \; , \tag{3}$$

with $s_W$ being the entropy per unit of horizon cross-sectional area. For a $D$-dimensional spacetime with metric $ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + \sum_{i,j=1}^{D-2} \sigma_{ij} dx_i dx_j$ , $dA = \sqrt{\sigma} dx_1 \ldots dx_{D-2}$ .

The actual Wald formula and how it comes about is briefly reviewed in Section 2. Meanwhile, it has since been shown by three of the current authors [5] how Wald's entropy density $s_W$ could also be extracted directly from the gravitational action. This, through a process of expanding the Lagrangian to quadratic order in the perturbations of the metric and evaluating it at the horizon. The density $s_W$ can then be identified with the coefficient of the

kinetic terms for the $r, t$-polarized gravitons $h_{rt}$. This coefficient measures the strength of the gravitational coupling for the same gravitons. The advantage of this method is that it identifies in a straightforward way the correct units in which the area of the horizon should be measured to give the correct value of the entropy. It also provides a way to decide which terms in the expanded action can contribute to the entropy.

Hence, one should always be able to obtain the Wald entropy by, first, expanding the Lagrangian around the background solution and, then, reading off the horizon value of the coefficient of the relevant kinetic terms; for instance, $\nabla^a h_{rt} \nabla_a h_{rt}$ . Equivalently and generally easier to implement, one may read the same coefficients off of terms like $\nabla^a \nabla_a h_{rt}$ in the linearized field equation.

For two-derivative theories of gravity, for instance, Einstein's and $F(\mathcal{R})$ theories, the procedure is as straightforward as just described. On the other hand, for theories with four or more derivatives, this seemingly simple process can become rather subtle. Our current motivations are to provide a well-defined prescription for identifying the kinetic terms and to better understand why Wald's formulation still works for these higher-derivative cases.

In this paper, we establish by explicit calculations that, for a completely generic theory of gravity, the kinetic contributions are indeed the only contributions to the Wald entropy. Further, we verify that the coefficients of these kinetic terms are always sufficient to reproduce Wald's formula for any number of derivatives that may appear explicitly or implicitly in Lagrangian. In the process we clarify the detailed properties of the Wald entropy that lead to such results. We then show that adding matter fields does not alter any of our results. We identify the correct form of the gravitational field equation that is suitable for calculations of the entropy via the linearized field equations.

The rest of the paper proceeds as follows. The next section briefly reviews the standard derivation of the Wald formula. The new material begins in Section 3, where we consider a general theory of gravity and expand the Lagrangian to second order in perturbations. Then, using basic properties of the metric, Riemann tensor and horizon generators, we distinguish

between terms that can possibly contribute to the entropy and those that cannot. We go on to verify that the surviving terms do indeed lead to an entropy in agreement with Wald's expression. In Section 4, the previous considerations are then extended to generic theories of both gravity and matter. Our attention turns, in Section 5, to the gravitational field equation for a generalized theory. In Section 6, some of specific models are used to illustrate our earlier analysis. Section 7 contains a summary and some concluding comments.

# 2 A brief review of the Wald Noether charge entropy

In this section, we recall the derivation of the Wald entropy, following Jacobson, Kang and Myers [6]. Our goal is to make the paper self-contained. For brevity, we skip over the many subtleties and caveats in the derivations.

One starts by varying a given Lagrangian density $L$ with respect to all the fields $\{\psi\}$, including the metric. In condensed notation (with all tensor indices suppressed),

$$\delta L \;=\; \mathcal{E}\cdot\delta\psi \;+\; d\theta\left(\delta\psi\right)\;, \tag{4}$$

where $\mathcal{E}=0$ are the equations of motion and the dot represents a summation over all fields and contractions of tensor indices. Also, $d$ denotes a total derivative, so that $\theta$ is a boundary term.

Let $\pounds_\xi$ be a Lie derivative acting along some vector field $\xi$. Then, given the diffeomorphism invariance of the theory, $\delta_\xi\psi=\pounds_\xi\psi$ and $\delta_\xi L=\pounds_\xi L=d\left(\xi\cdot L\right)$. These and Eq. (4) can be used to identify the associated Noether current $J_\xi$,

$$J_\xi \;=\; \theta\left(\pounds_\xi\psi\right)-\xi\cdot L\;. \tag{5}$$

The point being that $dJ_\xi=0$ when $\mathcal{E}=0$, and so there must be an associated "potential" $Q_\xi$ such that $J_\xi=dQ_\xi$. Now, if $D$ is the dimension of the spacetime and $\mathcal{S}$ is a $D-1$ hypersurface with a $D-2$ spacelike boundary $\partial\mathcal{S}$, then

$$\int_{\mathcal{S}} J_\xi \;=\; \int_{\partial\mathcal{S}} Q_\xi \tag{6}$$

is the associated Noether charge.

Wald showed [3] and later proved rigorously [4] that the black hole first law (2) is satisfied when the entropy is defined in terms of a specific Noether charge. Choosing the surface $S$ as the horizon $\mathcal{H}$ and the vector field $\xi$ as the horizon Killing vector $\chi$ (with its surface gravity normalized to unity [1]), Wald identified the entropy as

$$S_W \;\equiv\; 2\pi \oint_{\mathcal{H}} Q_\chi \;. \tag{7}$$

Since $\chi = 0$ on the horizon, [2] the right-most term in Eq. (5) does not contribute to $S_W$.

One of the main advantages of Wald's formula is the simplicity of $Q$. To understand this, let us start with the Killing identity [7]

$$\nabla_c \nabla_a \chi_b \;=\; -\mathcal{R}_{abcd}\chi^d \;, \tag{8}$$

where $\mathcal{R}_{abcd}$ is the Riemann tensor. After repeated applications of this relation, the most general form of the integrand in Eq. (7) can be expressed as

$$Q^{ab}\epsilon_{ab} \;=\; \left[\mathcal{B}^{ab}{}_c \chi^c + \mathcal{C}^{ab}{}_{cd}\nabla^c\chi^d\right]\epsilon_{ab} \;, \tag{9}$$

where $\mathcal{B}_{abc}$ and $\mathcal{C}_{abcd}$ are theory-dependent background tensors, while $\epsilon_{ab} \equiv \nabla_a \chi_b$ is the binormal vector for the horizon. For future reference,

$$\epsilon_{ab} \;=\; -\epsilon_{ba} \tag{10}$$

From the definition of $\epsilon_{ab}$ and since $\chi^a = 0$ (on $\mathcal{H}$), it follows that the integrand simplifies to

$$Q^{ab}\epsilon_{ab} = \mathcal{C}^{abcd}\epsilon_{ab}\epsilon_{cd} \tag{11}$$

or

$$S_W \;=\; 2\pi \oint_{\mathcal{H}} \mathcal{C}^{abcd}\epsilon_{ab}\epsilon_{cd} dA \;. \tag{12}$$

[1]This particular normalization for the Killing vector will be assumed throughout the paper.

[2]More accurately, at the bifurcation surface of the horizon. This is one of the many caveats that are dealt with in [6].

We continue to sketch the analysis of [6] for theories with a Lagrangian $\mathcal{L} = \mathcal{L}[g_{ab}, \mathcal{R}_{abcd}]$ .

First, the variation of the density $L = \sqrt{-g}\mathcal{L}$ is found to yield

$$\delta L \;=\; -2\nabla_a \left(\mathcal{X}^{abcd}\nabla_c \delta g_{bd}\sqrt{-g}\right) \;+\; \cdots \;, \tag{13}$$

where dots are meant as terms that end up being irrelevant to the Wald entropy and

$$\mathcal{X}^{abcd} \;\equiv\; \frac{\partial \mathcal{L}}{\partial \mathcal{R}_{abcd}} \;. \tag{14}$$

It follows from Eq. (4) that Eq. (13) leads to

$$\theta \;=\; -2n_a \mathcal{X}^{abcd}\nabla_c \delta g_{bd}\sqrt{\gamma} \;+\; \cdots \;, \tag{15}$$

where $n^a$ is the unit normal vector and $\gamma_{ab}$ is the induced metric for the chosen surface $\mathcal{S}$.

For an arbitrary diffeomorphism $\delta g_{ab} \;=\; \nabla_a \xi_b + \nabla_b \xi_a$ , the associated Noether current is then equal to

$$J \;=\; -2\nabla_a \left(\mathcal{X}^{abcd}\nabla_c \left(\nabla_b \xi_a + \nabla_a \xi_b\right) n_a \sqrt{h}\right) \;+\; \cdots \;. \tag{16}$$

Let us now specialize to the horizon $\mathcal{S} \to \mathcal{H}$ and (normalized) Killing vector $\xi^a \to \chi^a$ ; so that $n_a\sqrt{h} \to \epsilon_a \sqrt{\sigma}$ with $\epsilon_a \equiv \epsilon_{ab}\chi^b$ . Then, using the symmetries of $\mathcal{X}^{abcd}$ (inherited from $\mathcal{R}^{abcd}$) along with Eq. (10), one can eventually translate Eq. (16) into

$$J \;=\; -2\nabla_b \left(\mathcal{X}^{abcd}\nabla_c \chi_d \epsilon_a \sqrt{\sigma}\right) \;+\; \cdots \;. \tag{17}$$

In this form, the potential is

$$Q \;=\; -\mathcal{X}^{abcd}\epsilon_{ab}\epsilon_{cd}\sqrt{\sigma} \;+\; \cdots \;, \tag{18}$$

and so

$$S_W \;=\; -2\pi \oint_{\mathcal{H}} \mathcal{X}^{abcd}\epsilon_{ab}\epsilon_{cd} dA \;, \tag{19}$$

with $\mathcal{X}^{abcd}$ defined in Eq. (14).

# 3 Evaluating the Wald entropy

The main goal of this section is to establish our claims that only the kinetic terms for the $h_{rt}$ gravitons can contribute to the Wald formula and that the entropy can be deduced simply by reading off their coefficients. We accomplish this for a generic theory by, first, identifying all possible kinetic terms in the quadratic expansion of the Lagrangian and, then, verifying that the coefficients of the relevant terms produce a result that agrees with Wald's expression. Along the way, we find that the contributions to the Wald entropy arise from a specific class of terms in the quadratically expanded Lagrangian, those terms that contain a second-order expansion of the Riemann tensor.

## 3.1 Preliminaries

We will begin with a pure gravitational theory. The Lagrangian for such a theory can be expressed in terms of the metric, the Riemann tensor and its symmetrized covariant derivatives,

$$\mathcal{L} \;=\; \mathcal{L}\left[g_{ab}, \mathcal{R}_{abcd}, \nabla_{a_1}\mathcal{R}_{abcd}, \nabla_{(a_1}\nabla_{a_2)}\mathcal{R}_{abcd}, \ldots\right] \;, \tag{20}$$

where the ellipsis denote increasing numbers of symmetrized covariant derivatives acting on the Riemann tensor. The derivatives can be expressed in such a symmetrized form, as any anti-symmetric combination can be converted into a Riemann tensor.

Our objective is to expand the Lagrangian density $\sqrt{-g}\mathcal{L}$ to second order in the metric perturbations, $h_{ab} = g_{ab} - g^{(0)}_{ab}$ , and then isolate the two-derivative terms which we call "kinetic terms". We should also consider terms that have four or more derivatives, as clarified below. We will show that any kinetic term on the horizon can be expressed as

$$\left[\mathcal{A}^{abcd}\right]^{(0)} \overline{\nabla}^e h_{ab} \overline{\nabla}_e h_{cd} \;. \tag{21}$$

Here and in what follows, $\mathcal{A}^{a_1 a_2 \cdots}$ represents an arbitrary tensor built out of the Riemann tensor, its symmetrized derivatives and the metric. A numeric superscript on a tensor denotes its order in $h$'s and $\overline{\nabla}$ is a zeroth-order covariant derivative.

We can prove that a kinetic term is always of the form (21), which is a four-index tensor with the two derivatives contracted with each other. Let us begin with the most general term carrying exactly two gravitons and exactly two derivatives. Repeatedly integrating by parts and discarding surface terms and "mass terms" which have no derivatives acting on a graviton, we eventually arrive at

$$\left[\widetilde{\mathcal{A}}^{abcdef}\right]^{(0)} \overline{\nabla}_a h_{bc} \overline{\nabla}_d h_{ef} \ . \tag{22}$$

Here, the background tensor contains no explicit derivatives and is denoted by $\widetilde{\mathcal{A}}$.

Given our Lagrangian, the background tensor $\left[\widetilde{\mathcal{A}}^{abcdef}\right]^{(0)}$ is built out of the tensors $\left[g^{ab}\right]^{(0)}$ and

$$\left[R^{abcd}\right]^{(0)} \propto \left[g^{ac}g^{bd} - g^{ad}g^{bc}\right]^{(0)} \ . \tag{23}$$

The last expression follows from the fact that, when evaluated on a stationary horizon, any symmetric tensor $\mathcal{A}^{ab}_{sym}$ can be expressed as (see Section 5 of [8])

$$\mathcal{A}^{ab}_{sym} \ = \ \mathcal{A} g^{ab} \ , \tag{24}$$

for some scalar $\mathcal{A}$. [3] The background Ricci tensor $\left[\mathcal{R}^{ab}\right]^{(0)}$ is of this form. Then, given that $\mathcal{R}^{ac} = g_{bd}\mathcal{R}^{abcd}$ and $\mathcal{R}^{ad} = -g_{bc}\mathcal{R}^{abcd}$ , the form of Eq. (23) follows.

And so we have found that any index on $\left[\widetilde{\mathcal{A}}^{abcdef}\right]^{(0)}$ is associated with a metric tensor. Then, any index on

$$\overline{\nabla}_a h_{bc} \overline{\nabla}_d h_{ef} \tag{25}$$

must be contracted with one of the other five indices.

Let us now choose the transverse and traceless gauge for the gravitons, [4]

$$\overline{\nabla}_a h^a_{\ \ b}, \ h^a_{\ \ a} \ = \ 0 \ . \tag{26}$$

[3] This statement is valid on the horizon's bifurcation surface. However, as Wald's integral expression can be evaluated over an arbitrarily chosen cross-sectional slice [6], one can always calculate on the bifurcating slice without loss of generality.

[4] This choice is for convenience only, as the Wald entropy is gauge invariant.

So that contractions such as

$$\left[\widetilde{\mathcal{A}}^{abcd}\right]^{(0)}\overline{\nabla}_e h^e{}_a\overline{\nabla}_b h_{cd} \tag{27}$$

and

$$\left[\widetilde{\mathcal{A}}^{abcd}\right]^{(0)}\overline{\nabla}_e h_{ab}\overline{\nabla}_c h^e{}_d \tag{28}$$

vanish.

To show that the term (28) vanishes, we first argue that it must be of the form

$$\left[\widetilde{\mathcal{A}}^{bd}\right]^{(0)}\overline{\nabla}_e h^f{}_b\overline{\nabla}_f h^e{}_d\ , \tag{29}$$

otherwise a trace $h^a{}_a$ appears. Integrating by parts with $\overline{\nabla}_e$, we have

$$\left[\widetilde{\mathcal{A}}^{bd}\right]^{(0)}\overline{\nabla}_e h^f{}_b\overline{\nabla}_f h^e{}_d \ = \ -\overline{\nabla}_e\left(\left[\widetilde{\mathcal{A}}^{bd}\right]^{(0)}\right)h^f{}_b\overline{\nabla}_f h^e{}_d \ - \ \left[\widetilde{\mathcal{A}}^{bd}\right]^{(0)}h^f_b\overline{\nabla}_e\overline{\nabla}_f h^e{}_d\ . \tag{30}$$

All derivatives can be eliminated from the second term:

$$\begin{aligned}\overline{\nabla}_e\overline{\nabla}_f h^e{}_d \quad &= \quad \overline{\nabla}_f\overline{\nabla}_e h^e{}_d \ + \ \left[\overline{\nabla}_e,\overline{\nabla}_f\right]h^e{}_d \\ &= \quad \left[\mathcal{R}_{ef}{}^{e}{}_{a}\right]^{(0)}h^a{}_d \ + \ \left[\mathcal{R}_{efd}{}^{a}\right]^{(0)}h^e{}_a\ ,\end{aligned} \tag{31}$$

which follows from Eq. (26) and the standard identities relating commutators of derivatives to the Riemann tensor.

Integrating by parts $\overline{\nabla}_f$ in the first term in Eq. (30), we find

$$-\overline{\nabla}_e\left(\left[\widetilde{\mathcal{A}}^{bd}\right]^{(0)}\right)h^f{}_b\overline{\nabla}_f h^e{}_d = \overline{\nabla}_f\overline{\nabla}_e\left(\left[\widetilde{\mathcal{A}}^{bd}\right]^{(0)}\right)h^f{}_b h^e{}_d \ + \ \overline{\nabla}_e\left(\left[\widetilde{\mathcal{A}}^{bd}\right]^{(0)}\right)\left(\overline{\nabla}_f h^f{}_b\right)h^e{}_d\ , \tag{32}$$

so

$$-\overline{\nabla}_e\left(\left[\widetilde{\mathcal{A}}^{bd}\right]^{(0)}\right)h^f{}_b\overline{\nabla}_f h^e{}_d = \left[\mathcal{B}_{fe}{}^{bd}\right]^{(0)}h^f{}_b h^e{}_d\ ; \tag{33}$$

where, besides transversality, we have redefined the background tensor $\left[\mathcal{B}^{febd}\right]^{(0)} \equiv \overline{\nabla}^f\overline{\nabla}^e\left[\widetilde{\mathcal{A}}^{bd}\right]^{(0)}$ to emphasize that the above is really a mass term. Hence the term (28) is a mass term.

One can now see that the two $\overline{\nabla}$'s in a kinetic term must be contracted, leading to the claimed form in Eq. (21). But what about $h^2$ terms with greater numbers of derivatives? Terms having more than two $\overline{\nabla}$'s can also make kinetic contributions, as follows.

Derivatives have to be contracted in pairs since they cannot be contracted with a graviton index because of the gauge condition. Consequently, given a generic term with exactly $2n$ derivatives

$$\left[\widetilde{\mathcal{B}}^{a_1 a_2 \dots a_{2n-1} a_{2n}}\right]^{(0)} \overline{\nabla}_{a_1} \overline{\nabla}_{a_2} \dots \overline{\nabla}_{a_{2n-1}} \overline{\nabla}_{a_{2n}} \left( \left[\widetilde{\mathcal{A}}^{abcd}\right]^{(0)} h_{ab} h_{cd} \right) \,, \tag{34}$$

one can use integration by parts, commutator relations like

$$\left[\overline{\nabla}_a, \overline{\nabla}_b\right] h_{cd} \;=\; \left[\mathcal{R}_{abc}{}^{e}\right]^{(0)} h_{ed} + \left[\mathcal{R}_{abd}{}^{e}\right]^{(0)} h_{ce} \,, \tag{35}$$

symmetry properties of the background such as

$$\left[\mathcal{R}^{abcd}\right]^{(0)} \;=\; -\left[\mathcal{R}^{bacd}\right]^{(0)} \;=\; -\left[\mathcal{R}^{abdc}\right]^{(0)} \;=\; \left[\mathcal{R}^{dcba}\right]^{(0)} \tag{36}$$

$$\;=\; -\left[\mathcal{R}^{cabd}\right]^{(0)} \;-\; \left[\mathcal{R}^{bcad}\right]^{(0)} \tag{37}$$

and the gauge conditions (26), to manipulate the expression into a power series in $\Box \equiv \overline{\nabla}_e \overline{\nabla}^e$ acting on the gravitons. (Similar to our manipulation of Eq. (28) into a series terminating at $\Box^0$.) That is,

$$\sum_{j=0}^{n} \left[\mathcal{A}^{abcd}\right]_j^{(0)} h_{ab} \Box^j h_{cd} \,. \tag{38}$$

For instance, a term initially containing four derivatives and two gravitons can, after sufficient manipulations, be reduced to a combination of three forms:

$$\left[\widetilde{\mathcal{A}}^{abcd}\right]_2^{(0)} h_{ab} \Box^2 h_{cd} \,, \left[\mathcal{A}^{abcd}\right]_1^{(0)} h_{ab} \Box h_{cd} \,, \left[\mathcal{A}^{abcd}\right]_0^{(0)} h_{ab} h_{cd} \,. \tag{39}$$

This outcome follows from our earlier discussion, where it was shown that the background geometry can only act on gravitons so as to contract indices *or* yield pairs of contracted derivatives. The essential point is that higher-derivative terms can contribute to the kinetic terms and therefore to the Wald entropy, provided that all but two of the derivatives act on the background tensors.

### 3.2 Expanding the Lagrangian

Let us now begin the formal calculation by writing down the second-order expansion of $\sqrt{-g}\mathcal{L}$:

$$\delta\hat{\mathcal{L}}^{(2)} = \left[\frac{\partial\left(\sqrt{-g}\mathcal{L}\right)}{\partial g_{ab}}\frac{\delta g_{ab}}{\sqrt{-g}}\right]^{(2)} + \left[\frac{\partial\mathcal{L}}{\partial\mathcal{R}_{abcd}}\delta\mathcal{R}_{abcd}\right]^{(2)} + \left[\frac{\partial\mathcal{L}}{\partial[\nabla_{a_1}\mathcal{R}_{abcd}]}\delta\nabla_{a_1}\mathcal{R}_{abcd}\right]^{(2)}$$
$$+ \left[\frac{\partial\mathcal{L}}{\partial[\nabla_{(a_1}\nabla_{a_2)}\mathcal{R}_{abcd}]}\delta\nabla_{(a_1}\nabla_{a_2)}\mathcal{R}_{abcd}\right]^{(2)} + \ldots\,, \tag{40}$$

where

$$\delta\hat{\mathcal{L}} \equiv \frac{\delta\left[\sqrt{-g}\mathcal{L}\right]}{\sqrt{-g}} \tag{41}$$

and the ellipsis has now be used to denote variations with respect to ever-increasing numbers of symmetrized derivatives.

To proceed, we follow an iterative procedure that was first laid out in the third section of [4]. The basic idea is that a term like

$$\left[\frac{\partial\mathcal{L}}{\partial[\nabla_{(a_1}\ldots\nabla_{a_j)}\mathcal{R}_{abcd}]}\delta\nabla_{(a_1}\ldots\nabla_{a_j)}\mathcal{R}_{abcd}\right]^{(2)} \tag{42}$$

can always be re-expressed as

$$\left[\frac{\partial\mathcal{L}}{\partial[\nabla_{(a_1}\ldots\nabla_{a_j)}\mathcal{R}_{abcd}]}\nabla_{(a_1}\delta\nabla_{a_2}\ldots\nabla_{a_j)}\mathcal{R}_{abcd}\right]^{(2)} \tag{43}$$

plus terms that are proportional to $\nabla_{a_1}\delta g$. Then, integrating everything by parts, one has (up to surface terms)

$$-\left[\nabla_{(a_1}\left(\frac{\partial\mathcal{L}}{\partial[\nabla_{(a_1}\ldots\nabla_{a_j)}\mathcal{R}_{abcd}]}\right)\delta\nabla_{a_2}\ldots\nabla_{a_j)}\mathcal{R}_{abcd}\right]^{(2)} \tag{44}$$

plus terms that are proportional to $\delta g$. One can repeat this process $j-1$ more times until obtaining

$$(-)^j\left[\nabla_{(a_j}\ldots\nabla_{a_1)}\left(\frac{\partial\mathcal{L}}{\partial[\nabla_{(a_1}\ldots\nabla_{a_j)}\mathcal{R}_{abcd}]}\right)\delta\mathcal{R}_{abcd}\right]^{(2)}\,, \tag{45}$$

along with a collection of terms that are proportional to $\delta g$ (as well as surface terms).

Consequently, we can reorganize the expansion (40) as follows (up to surface terms):

$$\delta\hat{\mathcal{L}}^{(2)} = \left[\mathcal{W}^{ab}\delta g_{ab}\right]^{(2)} + \left[\mathcal{X}^{abcd}\delta\mathcal{R}_{abcd}\right]^{(2)}\,, \tag{46}$$

where $\mathcal{W}^{ab}$ is some tensorial function of the geometry (with its precise form being irrelevant to what follows [5]) and

$$\mathcal{X}^{abcd} \equiv \frac{\partial \mathcal{L}}{\partial \mathcal{R}_{abcd}} - \nabla_{a_1}\left(\frac{\partial \mathcal{L}}{\partial[\nabla_{a_1}\mathcal{R}_{abcd}]}\right) + \nabla_{(a_1}\nabla_{a_2)}\left(\frac{\partial \mathcal{L}}{\partial[\nabla_{(a_1}\nabla_{a_2)}\mathcal{R}_{abcd}]}\right) + \ldots \tag{47}$$

is a tensor with the same symmetry properties of the Riemann tensor. Notice that $\mathcal{X}^{abcd}$ is the generalized version of the same-named tensor in Eq. (14).

## 3.3 Isolating the kinetic terms

Now, since $\delta g_{ab} = g_{ab} - g^{(0)}_{ab}$, it follows that the complete non-expanded form of the $\mathcal{W}$ term in Eq. (46) contains a factor $g_{ab}$. Such a term can be dismissed, as integration by parts can be used to kill off any would-be kinetic contribution. [6]

To better understand this argument, suppose that we had the generic form

$$\left[\nabla_{(a_1}\cdots\nabla_{a_j)}h_{ab}\right]\mathcal{A}^{abcd;a_1\ldots a_j}(g^{(0)}_{cd} + h_{cd})\ . \tag{48}$$

Here, all expressions are regarded as full unexpanded expressions except where indicated. To create a kinetic term, it is then necessary to move derivatives (via integration by parts) until obtaining a combination of the form

$$\nabla_{a_j}h_{ab}\left[\nabla_{a_{j-1}}\cdots\nabla_{a_2}\mathcal{A}^{abcd;a_1\ldots a_j}\right]\nabla_{a_1}(g^{(0)}_{cd} + h_{cd})\ . \tag{49}$$

But such a term contains the vanishing factor $\nabla_{a_1}g_{cd} = 0$, and so the would-be kinetic term never has a chance to materialize.

[5] But note that, for $\mathcal{L} = \mathcal{L}[g_{ab}, \mathcal{R}_{abcd}]$ theories, $\mathcal{W}^{ab} = \partial\mathcal{L}/\partial g_{ab} + \frac{1}{2}g^{ab}\mathcal{L}$ .

[6] Let us emphasize that this is different than starting with, say, $\delta\mathcal{R}^{(2)} \sim \overline{\nabla}h\overline{\nabla}h + \cdots$ and then integrating by parts to come up with $h\overline{\nabla}\overline{\nabla}h$ . In this example, there may be an undifferentiated $h$, but it did not originate from a metric.

Having dismissed the metric variation, let us next focus on the $\delta\mathcal{R}_{abcd}$ contribution. Schematically, this goes as $\delta\mathcal{R} = \nabla\delta\Gamma + \delta\Gamma\delta\Gamma$ , with $\delta\Gamma = \nabla h$ . More precisely,

$$\delta\left[\Gamma^{a}_{bc}\right]^{(1)} \;=\; \frac{1}{2}\left[\overline{\nabla}_b h^{a}{}_{c} + \overline{\nabla}_c h^{a}{}_{b} - \overline{\nabla}^{a} h_{bc}\right] \;, \tag{50}$$

from which one obtains

$$\delta\mathcal{R}^{(1)}_{abcd}[h] \;=\; \frac{1}{2}\left[\overline{\nabla}_c\overline{\nabla}_b h_{ad} \;+\; \overline{\nabla}_d\overline{\nabla}_a h_{bc} \;-\; \overline{\nabla}_d\overline{\nabla}_b h_{ac} \;-\; \overline{\nabla}_c\overline{\nabla}_a h_{bd}\right] \tag{51}$$

and

$$\begin{aligned}\delta\mathcal{R}^{(2)}_{abcd} \;=\; \frac{1}{4}\Big[&\overline{\nabla}_c h_{ea}\overline{\nabla}_d h^{e}{}_{b} + \overline{\nabla}_c h_{ea}\overline{\nabla}_b h^{e}{}_{d} - \overline{\nabla}_e h_{ca}\overline{\nabla}^{e} h_{db} - \overline{\nabla}_a h_{ce}\overline{\nabla}_d h^{e}{}_{b} \\ -\overline{\nabla}_a h_{ce}\overline{\nabla}_b h^{e}{}_{d} - \overline{\nabla}_c h_{ea}\overline{\nabla}^{e} h_{db} &+ \overline{\nabla}_e h_{ca}\overline{\nabla}_d h^{e}{}_{b} + \overline{\nabla}_e h_{ca}\overline{\nabla}_b h^{e}{}_{d} + \overline{\nabla}_a h_{ce}\overline{\nabla}^{e} h_{db}\Big] \\ &-\; \{c \longleftrightarrow d\} \;. \end{aligned} \tag{52}$$

From these expansions, it can be deduced that

$$\begin{aligned}\delta\hat{\mathcal{L}}^{(2)}_{k} \;&=\; \left[\mathcal{X}^{abcd}\right]^{(0)}\delta\mathcal{R}^{(2)}_{abcd} \;+\; \left[\mathcal{X}^{abcd}\right]^{(1)}\delta\mathcal{R}^{(1)}_{abcd} \\ &=\; \frac{1}{4}\left[\mathcal{X}^{abcd}\right]^{(0)}\left(\overline{\nabla}_c h_{ea}\overline{\nabla}_d h^{e}{}_{b} + \cdots\right) \;+\; \frac{1}{2}\left[\mathcal{X}^{abcd}\right]^{(1)}\left(\overline{\nabla}_c\overline{\nabla}_b h_{ad} + \cdots\right) \;,\end{aligned} \tag{53}$$

where the subscript $k$ indicates that we only intend to retain the kinetic contributions and the ellipses denote the various permutations of the displayed indices. The first term on the right-hand side can clearly be kinetic and is what would normally be attributed to the Wald entropy. The question then is what becomes of the second term?

To make sense of the second term, it is necessary to expand the tensor $\mathcal{X}^{abcd}$ to first order. Following the same iterative procedure as before, we have

$$\left[\mathcal{X}^{abcd}\right]^{(1)} \;=\; \left[\mathcal{Z}^{abcd;ef}\right]^{(0)}\delta g^{(1)}_{ef} \;+\; \left[\mathcal{Y}^{abcd}_{pqrs}\right]^{(0)}\delta\mathcal{R}^{(1)}_{pqrs} \;, \tag{54}$$

where $\mathcal{Z}^{abcd;ef}$ is a tensor akin to $\mathcal{W}^{ab}$ and

$$\begin{aligned}\mathcal{Y}^{abcd}_{pqrs} \;\equiv\; \frac{\partial\mathcal{X}^{abcd}}{\partial\mathcal{R}_{pqrs}} \;&-\; \nabla_{a_1}\left(\frac{\partial\mathcal{X}^{abcd}}{\partial[\nabla_{a_1}\mathcal{R}_{pqrs}]}\right) \\ &+\; \nabla_{(a_1}\nabla_{a_2)}\left(\frac{\partial\mathcal{X}^{abcd}}{\partial[\nabla_{(a_1}\nabla_{a_2)}\mathcal{R}_{pqrs}]}\right) \;+\; \ldots\end{aligned} \tag{55}$$

is a tensor that is "two-fold" Riemann symmetric.

Since $\delta g_{ef} = g_{ef} - g^{(0)}_{ef}$ , we can, as previously discussed, neglect the first term on the right-hand side of Eq. (54). Also recalling Eq. (51) for $\delta\mathcal{R}^{(1)}$, we have

$$\left[\mathcal{X}^{abcd}\right]^{(1)} = \frac{1}{2}\left[\mathcal{Y}^{abcd}_{pqrs}\right]^{(0)}\left[\overline{\nabla}_r\overline{\nabla}_q h_{ps} + \overline{\nabla}_s\overline{\nabla}_p h_{qr} - \overline{\nabla}_s\overline{\nabla}_q h_{pr} - \overline{\nabla}_r\overline{\nabla}_p h_{qs}\right] . \tag{56}$$

The insertion of Eq. (56) into Eq. (54) then yields

$$\begin{aligned}\delta\hat{\mathcal{L}}^{(2)}_k &= \frac{1}{4}\left[\mathcal{X}^{abcd}\right]^{(0)}\left(\overline{\nabla}_c h_{ea}\overline{\nabla}_d h^e_b + \cdots\right) \\ &+ \frac{1}{4}\left[\mathcal{Y}^{abcd}_{pqrs}\right]^{(0)}\left(\overline{\nabla}_c\overline{\nabla}_b h_{ad}\,\overline{\nabla}_r\overline{\nabla}_q h_{ps} + \cdots\right) .\end{aligned} \tag{57}$$

As already mentioned, a four-$\overline{\nabla}$ term might still make a kinetic contribution, so that the second term can not be dismissed. Nevertheless, we will now proceed to demonstrate that such a term can not contribute to the Wald entropy, simply because this is actually a mass term.

In the Wald-entropy prescription [3, 4, 6] the relevant metric perturbations are from the restricted class $h_{ab}|_{\mathcal{H}\,;\,\{a,b\}=\{r,t\}}$ and

$$h_{ab} = \nabla_a\chi_b + \nabla_b\chi_a , \tag{58}$$

where $\chi_a$ is a vector field that (at least) limits to the Killing vector on the horizon. Actually, as mentioned in Section 1, the Killing vector has been normalized such that

$$\epsilon_{ab} = \nabla_a\chi_b , \tag{59}$$

is the horizon binormal, for which

$$\epsilon_{ab} \neq 0 \quad \text{iff} \quad \{a \neq b\} = \{r,t\} ; \tag{60}$$

and so only the off-diagonal elements $h_{rt}$ are relevant.

Let us now consider a term such as $\nabla_c\nabla_b\nabla_a\chi_d$ . One can use the Killing identity (Eq. (8)) to rewrite this as

$$\begin{aligned}\nabla_c\nabla_b\nabla_a\chi_d &= \nabla_c\left[\mathcal{R}_{dabe}\chi^e\right] \\ &= \mathcal{R}_{dabe}\nabla_c\chi^e ,\end{aligned} \tag{61}$$

where the second line is a consequence of $\chi^a = 0$ on the horizon. This outcome leads us to, for instance,

$$\begin{aligned}
\overline{\nabla}_c \overline{\nabla}_b h_{ad} &= \mathcal{R}^{(0)}_{dabe} \overline{\nabla}_c \chi^e + \mathcal{R}^{(0)}_{adbe} \overline{\nabla}_c \chi^e \\
&= \mathcal{R}^{(0)}_{dabe} \left[ \overline{\nabla}_c \chi^e + \overline{\nabla}^e \chi_c \right] \\
&= \mathcal{R}^{(0)}_{dabe} h^e_c \;,
\end{aligned} \tag{62}$$

where Eq. (58) has been used twice and the second line comes about from the anti-symmetries of both the Riemann tensor and the horizon binormal vector; *cf*, Eq. (10).

It now follows that the term proportional to $\mathcal{Y}$ in Eq. (57) can equivalently be written as

$$\frac{1}{4} \left[ \mathcal{Y}^{abcd}_{pqrs} \right]^{(0)} \left( \mathcal{R}^{(0)}_{dabe} h^e_c \, \mathcal{R}^{(0)}_{spqw} h^w_r \;+\; \cdots \right) \;; \tag{63}$$

so that any such term is a mass term as promised. [7]

Hence, the kinetic contribution and the Wald entropy is determined strictly by the first part of Eq. (57). Following the result from Eq. (23) of [5] (which follows from Eq. (52) and the symmetry properties of $\left[\mathcal{X}^{abcd}\right]^{(0)}$)

$$\left[\mathcal{X}^{abcd}\right]^{(0)} \delta\mathcal{R}^{(2)}_{abcd} \;=\; \frac{1}{2} \left[\mathcal{X}^{abcd}\right]^{(0)} \left( \overline{\nabla}^e h_{bc} \overline{\nabla}_e h_{ad} + 2\overline{\nabla}^e h_{ac} \overline{\nabla}_b h_{de} \right) \tag{64}$$

and recognizing that the second term can be gauged away, we arrive at

$$\delta\hat{\mathcal{L}}^{(2)}_k \;=\; \frac{1}{2} \left[\mathcal{X}^{abcd}\right]^{(0)} \; \overline{\nabla}^e h_{bc} \overline{\nabla}_e h_{ad} \;. \tag{65}$$

We can be even more precise by remembering that the relevant perturbations are sourced strictly by the off-diagonal elements in the $\{r, t\}$ sector of gravitons. Hence,

$$\delta\hat{\mathcal{L}}^{(2)}_k \;=\; \frac{1}{2} \left[\mathcal{X}^{rt}{}_{rt}\right]^{(0)} \sum_{a\neq b}^{r,t} \overline{\nabla}^e h_{ab} \overline{\nabla}_e h^{ab} \;. \tag{66}$$

[7] If the purpose is to calculate the Wald entropy, then one cannot "cheat" by having the Killing identity preceded by an integration by parts. That is, one is not permitted to turn $\delta\mathcal{R}^{(2)} \sim \nabla h \nabla h$ into $h \nabla\nabla h \sim h\mathcal{R}h$ and then argue that this is a mass term.

### 3.4 Interpretations

Our claim is that all information about the Wald entropy is (up to normalization [8]) encoded in the horizon value of the single tensorial component $[\mathcal{X}^{rt}{}_{rt}]^{(0)}$,

$$s_W \;=\; \mathcal{C}\left[\mathcal{X}^{abcd}\right]^{(0)}\epsilon_{ab}\epsilon_{cd}\;, \tag{67}$$

where $s_W$ is the "Wald entropy density" (or entropy per unit of horizon cross-sectional area), $\mathcal{C}$ is a universal normalization constant and we have made use of Eq. (60) to express the result in terms of the horizon binormal vectors.

To compare, let us recall that the Wald entropy goes as

$$S_W \;=\; \oint_{\mathcal{H}} s_W dA\;, \tag{68}$$

where $dA$ denotes an area element for a cross-section of the horizon $\mathcal{H}$. The density $s_W$ goes (in our notation) as

$$s_W \;=\; -2\pi\left[\mathcal{X}^{abcd}\right]^{(0)}\epsilon_{ab}\epsilon_{cd}\;. \tag{69}$$

And so the Wald formula agrees with our expression (67), with the normalization now fixed at $\mathcal{C} = -2\pi$ .

Let us further clarify the relation between the Wald entropy and the coefficients of the kinetic graviton terms. We recall that, for the derivations of the Wald entropy [3, 4, 6], the idea was to start with the linearized field equation

$$\frac{1}{\sqrt{-g}}\left[\frac{\partial\left(\sqrt{-g}\mathcal{L}\right)}{\partial g^{ab}}\right]^{(0)} h_{ab} \;=\; 0 \tag{70}$$

and reduce this to a boundary term over a cross-section of the horizon. Following this path, one ends up with various terms of the generic form

$$\left[\mathcal{A}^{a_1\ldots a_j;ab}\right]^{(0)}\overline{\nabla}_{a_1}\ldots\overline{\nabla}_{a_j}h_{ab}\;. \tag{71}$$

[8] The correct normalization can always be uniquely fixed by the Einstein Lagrangian. In this case, $\mathcal{X}^{ab}{}_{cd} = \frac{1}{2}\left[g^a_c g^b_d - g^a_d g^b_c\right]$ , and so $[\mathcal{X}^{rt}{}_{rt}]^{(0)} = +\frac{1}{2}$ .

We now recall from the previous subsection that the relevant gravitons can, when pulled back to the horizon, be exchanged for $\epsilon_{ab}$'s or $\overline{\nabla}_a \chi_b$ 's (see Eqs. (58,59)). Meaning that the Killing identity (8) can be used to reduce the number of derivatives when $j > 0$ and, since the Killing vector vanishes on the horizon, any of these terms can be cast into the form

$$\left[\mathcal{A}^{ab}\right]^{(0)} h_{ab} \,. \tag{72}$$

Let us next consider the process of going from the original volume integral to a surface integral over the horizon and, subsequently, to a closed integral over a horizon cross-section. One is then required to apply Gauss' theorem twice. Starting with the integrated form of Eq. (72) and following this route backwards, we have

$$\begin{aligned} \oint_{\mathcal{H}} \left[\mathcal{A}^{ab}\right]^{(0)} h_{ab} dA &= \int_{\mathcal{H}} \left[\partial_\lambda\right]^c \overline{\nabla}_c \left( \left[\mathcal{A}^{ab}\right]^{(0)} h_{ab} \right) d\lambda dA \\ &= \int_{\mathcal{M}} \Box \left( \left[\mathcal{A}^{ab}\right]^{(0)} h_{ab} \right) \sqrt{-g_{tt} g_{rr}} dr dt dA \,, \end{aligned} \tag{73}$$

with $\lambda$ being the affine parameter for the horizon and $\mathcal{M}$ the exterior spacetime. Although the first equality is trivial, the second equality is quite complicated, as sensitive limiting procedures are required to translate coordinates and geometric quantities from the null horizon to a timelike "stretched horizon". The final form also assumes that the graviton and background do not depend on the "non-radial" spatial coordinates $x_1, x_2 \ldots x_{D-2}$. Yet, the underlying message is clear: Any contribution to the Wald entropy must necessarily come about from terms in the linearized field equation (or, equivalently, the quadratic action) carrying two derivatives. Our analysis explicitly establishes this connection and also identifies the origin of the kinetic terms.

As observed elsewhere [5, 9], the Wald entropy can be generalized to other types of gravitational couplings by a different choice of polarization for the gravitons. For instance, the shear viscosity of a black brane is determined by the kinetic coefficient of the $h_{xy}$ gravitons, where $x$ and $y$ are transverse directions on the brane that are mutually orthogonal as well as orthogonal to the direction of propagation. On this basis, it had been conjectured [9] that

the shear viscosity $\eta$ could be determined in analogous fashion to the entropy; that is (*cf*, Eq. (67)),

$$\eta \;=\; \mathcal{C}_\eta \sum_{a \neq b}^{x,y} \left[\mathcal{X}^{abcd}\right]^{(0)} \tilde{\epsilon}_{ab}\tilde{\epsilon}_{cd} \;, \tag{74}$$

where $\tilde{\epsilon}_{ab}$ is a suitably defined binormal vector.

However, our previous use of the Killing vector is unique to the $h_{rt}$ gravitons on the horizon and, hence, unique to the Wald entropy. For this reason, the calculation of any other type of coupling (such as $\eta$) would generally involve the $\mathcal{Y}$ term in Eq. (57). This term only becomes relevant for six (or higher) derivative theories. This is because, for a two (four) derivative theory, $\mathcal{Y} = 0$ ( $\mathcal{Y} \sim gg$ ); and the would-be kinetic terms are either identically zero or effectively zero through integration by parts.

### 3.5 Summary

In this section we have verified our assertion that, for a theory of gravity with any number of derivatives, the kinetic terms for the $h_{rt}$ gravitons completely account for the Wald entropy. By expanding out the Lagrangian to second order in gravitational perturbations, we have established that, due to the Killing identity, the only contributing terms are those for which a single component of the Riemann tensor is responsible for both of the gravitons. This property is essential to the applicability of the Wald formula to higher-derivative gravitational theories. Indeed, analogue formulas for other types of graviton coupling would, as discussed above, break down at the six-derivative order.

## 4 Including matter

Our considerations have so far been limited to theories of gravity without matter. It is then natural to ask if the inclusion of matter fields could alter any of the results of the preceding section. We now address this question and demonstrate that, even for a general theory of gravity coupled to matter, all our previous conclusions remain valid.

## 4.1 Preliminaries

Let us now add matter fields, denoted collectively by $\psi$. Since anti-symmetric combinations of derivatives can always be replaced by $\mathcal{R}$'s, the Lagrangian is of the form

$$\mathcal{L} \;=\; \mathcal{L}\left[g_{ab}, \mathcal{R}_{abcd}, \nabla_{a_1}\mathcal{R}_{abcd}, \nabla_{(a_1}\nabla_{a_2)}\mathcal{R}_{abcd}, \ldots\;;\; \psi, \nabla_{a_1}\psi, \nabla_{(a_1}\nabla_{a_2)}\psi, \ldots\right] . \tag{75}$$

In principle, one should then add the following set of terms to the expansion of $\mathcal{L}^{(2)}$ in Eq. (40):

$$\left[\frac{\partial\mathcal{L}}{\partial\psi}\delta\psi\right]^{(2)} + \left[\frac{\partial\mathcal{L}}{\partial[\nabla_{a_1}\psi]}\delta\nabla_{a_1}\psi\right]^{(2)} + \left[\frac{\partial\mathcal{L}}{\partial[\nabla_{(a_1}\nabla_{a_2)}\psi]}\delta\nabla_{(a_1}\nabla_{a_2)}\psi\right]^{(2)} + \;\ldots . \tag{76}$$

However, for scalar (vector, tensor) matter fields, at least the first four (three, two) terms in the series will not contribute. To understand this point, let us consider the case of three derivatives or less acting on a scalar $\phi$. (Two derivatives acting on a vector and one, on a tensor would be equivalent situations.) As we have seen, a kinetic graviton term can only arise out of a variation of the Riemann tensor,

$$\delta\mathcal{R} \;\sim\; \nabla\delta\Gamma + \delta\Gamma\delta\Gamma \quad \text{where} \quad \delta\Gamma \sim \nabla h\;, \tag{77}$$

or out of variations of Christoffel symbols. But recall that only the $\delta\Gamma\delta\Gamma$ part of the Riemann variation contributes to the Wald entropy. Hence, a minimal requirement is having two covariant derivatives that act as Christoffel symbols.

Now, for a scalar field, three-derivative terms can be dismissed because, as already stressed in Subsection 2.1, derivatives can only emerge out of the background in pairs. The presence of a scalar field (unlike a vector and other odd-spin fields) cannot viably alter this outcome. This leaves the remaining possibility of

$$\mathcal{A}^{ab}\nabla_{(a}\nabla_{b)}\phi\;. \tag{78}$$

At a first glance, this seems to satisfy the minimal requirement. But, as the right-most derivative is required to act directly on $\phi$, the above expression reduces to

$$\mathcal{A}^{ab}\nabla_{(a}\partial_{b)}\phi\;, \tag{79}$$

and only one Christoffel symbol is available. It follows that such a term can only induce the variations $\delta g_{ab}$ and $\delta\phi$; thus disqualifying it as a kinetic contributor.

However, the story could change if the matter sector had a sufficiently large number of symmetrized derivatives. The minimum requirements being at least four derivatives with a scalar field, three with a vector or two with a tensor.

## 4.2 Four derivative terms

Let us first consider the simplest example of possible contributions:

$$\mathcal{L}_\phi \;=\; \phi g^{ab} g^{cd} \nabla_{(a} \nabla_b \nabla_c \nabla_{d)} \phi \;, \tag{80}$$

with $\phi$, again, a scalar matter field. [9] The variation of $\mathcal{L}_\phi$ with respect to $\nabla_{(a_1} \nabla_{a_2} \nabla_{a_3} \nabla_{a_4)} \phi$ will lead to a "candidate" kinetic term of the form

$$\left[\delta\mathcal{L}_\phi\right]^{(2)}_k \;=\; \left[\phi g^{ab} g^{cd}\right]^{(0)} \left[\delta \nabla_{(a} \nabla_b \nabla_c \nabla_{d)} \phi\right]^{(2)} \;. \tag{81}$$

A kinetic term might appear if any two of the derivatives act as a Christoffel symbol and the remaining two act directly on the scalar since such a combination leads to the the schematic form $\left[\delta\left(\Gamma\Gamma\nabla\nabla\phi\right)\right]^{(2)}$, and so $\Gamma[h]\Gamma[h]\overline{\nabla}\overline{\nabla}\phi \sim \overline{\nabla}\overline{\nabla}\phi\overline{\nabla}h\overline{\nabla}h$. Nonetheless, explicit calculations have indicated that the net kinetic term from this Lagrangian vanishes. This finding can be explained by the following observations: Even though $\mathcal{L}_\phi$ appears to have $4! = 24$ distinct terms, these can be joined into three types

$$\mathcal{L}_\phi \;=\; 8\phi\Box^2\phi + 8\phi\nabla^a\nabla^b\nabla_a\nabla_b\phi + 8\phi\nabla^a\Box\nabla_a\phi \;. \tag{82}$$

One can then use the commutation relations (*e.g.*, $[\nabla_a, \nabla_b]\nabla_c = \mathcal{R}_{abc}{}^{d}\nabla_d$ ) to iteratively convert the second and third type into the first. For instance, one of the 24 terms goes as

$$\begin{aligned}
\phi g^{ab} g^{cd} \nabla_a \nabla_c \nabla_b \nabla_d \phi \;&=\; \phi g^{ab} g^{cd} \nabla_a \nabla_b \nabla_c \nabla_d \phi \;+\; \phi g^{ab} g^{cd} \nabla_a \left[\nabla_c, \nabla_b\right] \nabla_d \phi \\
&=\; \phi\Box^2\phi \;+\; \phi g^{ab} g^{cd} \nabla_a \mathcal{R}_{cbd}{}^{e} \nabla_e \phi \;;
\end{aligned} \tag{83}$$

[9]The second scalar field in front is to prevent this Lagrangian from being a total derivative.

and similarly for the other 15 terms that are not initially of the $\phi\Box^2\phi$ type.

The end result of the just-described process is

$$\mathcal{L}_\phi \;=\; 24\phi\Box^2\phi + \phi g^{(ab}g^{cd)}\nabla_a\left(\mathcal{R}_{bcd}{}^{e}\,\nabla_e\phi\right) \;, \tag{84}$$

with symmetrized indices on the metrics. Although there (again) appears to be $4! = 24$ different terms in the Riemann part of the above expression, this really only contains 16 such terms. The reason being that 8 of the 24 terms are of the form

$$\phi g^{ad}g^{bc}\nabla_a\mathcal{R}_{bcd}{}^{e}\nabla_e\phi \;, \tag{85}$$

which is already identically vanishing through the contraction of the first two Riemann indices.

The rest of the Riemann part vanishes identically as well. To understand this, let us suppose that the index on the derivative $\nabla_a$ is fixed while the other 3 Riemann indices $bcd$ remain symmetrized. This leads to the following 6 terms (where we display only the Riemann tensor for brevity):

$$\mathcal{R}_{(bcd)}{}^{e} \;=\; \mathcal{R}_{bcd}{}^{e} + \mathcal{R}_{dbc}{}^{e} + \mathcal{R}_{cdb}{}^{e} + \mathcal{R}_{bdc}{}^{e} + \mathcal{R}_{cbd}{}^{e} + \mathcal{R}_{dcb}{}^{e} \;. \tag{86}$$

But, the first three terms sum to zero and, likewise, the latter three terms, due to the Jacobi identity; *cf*, Eq.(37). Then, since the sum total is (as we vary the index on $\nabla$) four such vanishing sets, the Riemann part of Eq. (84) is zero.

What is left to establish is that $\phi\left[\delta\Box^2\phi\right]^{(2)}$ similarly makes no kinetic contribution. This can, indeed, be verified with an explicit calculation but also follows from a simple argument. To show this, let us first recall that

$$\Box^2\phi \;=\; \frac{1}{\sqrt{-g}}\partial_a\left\{g^{ab}\sqrt{-g}\,\partial_b\left[\frac{1}{\sqrt{-g}}\partial_c\left(g^{cd}\sqrt{-g}\,\partial_d\phi\right)\right]\right\} \;. \tag{87}$$

After we disregard all kinetic contributions that can be gauged away, which include either derivatives acting on determinants or derivatives acting transversely, one finds that the sole potential contributor is

$$\Box^2\phi \;\to\; g^{ab}\left[\partial_a\partial_b g^{cd}\right]\left[\partial_c\partial_d\phi\right] \;. \tag{88}$$

However, even this term can still be gauged away through integration by parts, and so vanishes.

### 4.3 2n derivative terms

The same trend extends to any number of symmetrized derivatives. Again working with a scalar matter field $\phi$ for simplicity, we can justify this claim via the following arguments:

Let us begin by considering the Lagrangian

$$\mathcal{L}_{2n}(\phi) \ = \ \phi g^{a_1 a_2} \cdots g^{a_{2n-1}a_{2n}} \nabla_{(a_1} \nabla_{a_2} \dots \nabla_{a_{2n-1}} \nabla_{a_{2n})} \phi \ , \tag{89}$$

which contains $2n$ symmetrized derivatives. Similarly to the four-derivative case, $\mathcal{L}_{2n}(\phi)$ should decompose into the form

$$\phi \Box^n \phi \quad + \quad \phi \sum_{k=1}^{n-1} \left[ \mathcal{R}^{[k]} \nabla^{[2n-2k-1]} \right]^a \nabla_a \phi \ , \tag{90}$$

with the square bracket meant to represent a collection of $k$ (4-index) Riemann tensors followed by $2n-2k-1$ $\nabla$'s contracted in all possible ways. Such an arrangement is always possible given a sufficient number of a commutations of derivatives. [10]

Now, the key point is that any of the generated Riemann tensors arises due to a commutation of symmetrized $\nabla$'s, and so is of the basic form

$$\left[ \nabla_{(a_i}, \nabla_{a_j} \right] \nabla_{a_k)} \ \dots \ = \ \mathcal{R}_{(a_i a_j a_k)}{}^{e} \, \nabla_e \ \dots \ , \tag{91}$$

whereby three of the Riemann indices are explicitly symmetrized while the fourth index is summed over independently. So that, just like for the 4-derivative example (Eq. (86)) any Riemann tensor produces two sets of 3 terms with each vanishing due to the Jacobi identity.

Let us now focus on the $\phi \Box^n \phi$ term. Again disregarding kinetic contributions that can be gauged away, we are left with potential contributors of only the two basic forms,

$$\dots g^{ab} g^{cd} \partial_a h^{ef} \partial_c h^{ij} \dots \partial_b \partial_d \partial_e \partial_f \partial_i \partial_j \phi \quad \text{and} \quad \dots g^{ab} \partial_a h^{cd} \partial_b h^{ij} \dots \partial_c \partial_d \partial_i \partial_j \phi \ , \tag{92}$$

[10] To ensure the displayed ordering, one should commute derivatives from left to right; *i.e.*, opposite to the direction of Eq. (83).

with the dots indicating other irrelevant structure. The crucial point here is that the derivatives act only symmetrically on the gravitons, whereas a Riemann tensor is constructed out of anti-symmetric combinations of derivatives. Meaning that these terms are simply not capable of hiding a Riemann tensor.

### 4.4 Summary

Given a generalized theory of gravity, we have now shown that adding matter fields with any number of symmetrized covariant derivatives acting on them does not lead to any kinetic contributions beyond those already encountered in Section 3. The explicit calculations used scalar fields but can, as discussed, be generalized in a straightforward manner to any type of tensor field.

## 5 The generalized field equation

The calculation of the quadratic action for a generalized theory is often complicated. But one can rather use the linearized field equation as an equivalent but simpler means for extracting the Wald entropy. The compatibility of the two approaches follows from their equivalency up to total derivatives. An appropriate choice of surface term can always be used to cancel a total derivative and then, as explained in [6], any such boundary term does not contribute to the Wald formula.

The entropy should be extracted from terms of the schematic form $\nabla\nabla h$ in the field equation, subject to the various subtleties discussed. The field equation has already been presented (without a full derivation) in [4]. However, for practical calculations, we have identified a form of the equation differing by a sign from that of [4].

## 5.1 The linearized field equation

Let us presume, for simplicity in the presentation, that the matter fields $\psi$ are coupled only to the metric and any such terms have been collected separately in $\frac{1}{2}\mathcal{L}_{\psi}$. Then the gravitational field equation for the density

$$\mathcal{L}_{total} \;=\; \sqrt{-g}\left[\mathcal{L} + \frac{1}{2}\mathcal{L}_{\psi}\right] \tag{93}$$

is

$$\frac{\partial \mathcal{L}}{\partial g_{pq}}\delta g_{pq} \;+\; \frac{1}{2}g^{pq}\mathcal{L}\delta g_{pq} \;+\; \mathcal{X}^{abcd}\delta\mathcal{R}_{abcd}[\delta g_{pq}] \;+\; \mathcal{Q}^{pq}\delta g_{pq} \;=\; -T^{pq}\delta g_{pq}\;, \tag{94}$$

where

$$T^{pq} \;\equiv\; \frac{2}{\sqrt{g}}\frac{\partial\left[\sqrt{-g}\mathcal{L}_{\psi}\right]}{\partial g_{pq}}\;, \tag{95}$$

$\mathcal{X}^{abcd}$ is defined in Eq. (47) and the tensor $\mathcal{Q}^{pq}$ accounts for the extraneous terms in the iterative procedure of Section 2. (But, if $\mathcal{L}$ contains only anti-symmetrized combinations of derivatives, there would be no $\mathcal{Q}$ contribution.) Note that, had we varied with respect to the contravariant form of the metric $\delta g^{pq}$, then the stress tensor needs to be defined as

$$T_{pq} \;\equiv\; -\frac{2}{\sqrt{g}}\frac{\partial\left[\sqrt{-g}\mathcal{L}_{\psi}\right]}{\partial g^{pq}}\;. \tag{96}$$

Now linearizing and rearranging, we have

$$\left[\mathcal{X}^{abcd}\right]^{(0)}\delta\mathcal{R}^{(1)}_{abcd}[h_{pq}] \;=\; -\left[T^{pq} \;+\; \frac{\partial\mathcal{L}}{\partial g_{pq}} \;+\; \frac{1}{2}g^{pq}\mathcal{L} \;+\; \mathcal{Q}^{pq}\right]^{(0)} h_{pq}\;, \tag{97}$$

with the right-hand side being irrelevant to the Wald entropy.

We will thus focus on the left-hand side and recall the expansion for $\delta\mathcal{R}^{(1)}_{abcd}$ in Eq. (51). This expression and the symmetry properties of the background Riemann tensor (which are shared by $\left[\mathcal{X}^{abcd}\right]^{(0)}$) indicate that the left-hand side of Eq. (97) reduces to the sum of four equivalent terms. Denoting this sum as $\mathcal{G}$, we obtain

$$\begin{aligned}\mathcal{G} \;=\; \left[\mathcal{X}^{abcd}\right]^{(0)}\delta\mathcal{R}^{(1)}_{abcd}[h_{pq}] \;&=\; 2\left[\mathcal{X}^{apqb}\right]^{(0)}\overline{\nabla}_a\overline{\nabla}_b h_{pq} \\ &=\; \mathcal{G}^{pq}h_{pq}\;.\end{aligned} \tag{98}$$

For the rest of this section, it is implied that the zeroth-order geometry applies to all tensors besides the $h$'s.

To proceed, one considers separately the symmetric $\frac{1}{2}\{\nabla_a, \nabla_b\}$ and the anti-symmetric $\frac{1}{2}[\nabla_a, \nabla_b]$ combinations of the derivatives. Whereas the former is trivially handled with a double integration by parts, the latter is more complicated. Nonetheless, it is possible to convert $\{\nabla_a, \nabla_b\}$ into Riemann tensors. Then, through repeated application of the symmetries of both the $\mathcal{X}$ and $\mathcal{R}$ tensors, one obtains the expression in the second line of Eq. (98). What is left is to strip off the graviton and then linearize $\mathcal{G}_{pq}$.

Skipping over subtleties, we find that the complete field equation now goes as (*cf*, Eq. (97))

$$2\nabla_b \nabla_a \mathcal{X}^{apqb} \;-\; \mathcal{X}^{abcp}\mathcal{R}_{abc}{}^{q} \;+\; \frac{1}{2}g^{pq}\mathcal{L} \;=\; -T^{pq} \;, \tag{99}$$

where we have assumed that $\mathcal{L} = \mathcal{L}\left[\mathcal{R}_{abcd}\right]$ for simplicity. This is in contrast to the relative positive sign between the first terms in Eq. (99) in [4].

## 5.2 Simple examples

To clarify this process, let us recall the Einstein Lagrangian, $\mathcal{L}_{Ein} = \mathcal{R}$, regarded as independent of the metric. Plugging this into Eq. (47), we find that

$$\mathcal{X}^{abcd}_{Ein} \;=\; \frac{1}{2}\left[g^{ac}g^{bd} - g^{ad}g^{bc}\right] \;, \tag{100}$$

and the tensor $\mathcal{G}^{pq}$ of Eq. (98) then reduces to

$$\mathcal{G}^{pq}_{Ein} = -\mathcal{R}^{pq} \;. \tag{101}$$

This is indeed the correct form, as substituting back into the field equation (99), we obtain

$$\mathcal{R}^{pq} \;-\frac{1}{2}g^{pq}\mathcal{R} \;=\; T^{pq} \;. \tag{102}$$

Let us further confirm the consistency of Eq. (98) by starting with an $\mathcal{L} = F(\mathcal{R})$ theory of gravity ( $\mathcal{R} = g^{ac}g^{bd}\mathcal{R}_{abcd}$ ) and following the standard procedure,

$$\partial_{\mathcal{R}}F(\mathcal{R})\left[\mathcal{R}_{pq}\delta g^{pq} \;+\; g^{pq}\delta\mathcal{R}_{pq}\right] \;-\; \frac{1}{2}F(\mathcal{R})g_{pq}\delta g^{pq} \;=\; T_{pq}\delta g^{pq} \;. \tag{103}$$

To handle the Ricci variation, one can use a contracted form of Eq. (51) and then integrate by parts (twice) to free up the graviton. In this way, one ultimately finds that (*e.g.*, [10])

$$\left[\mathcal{R}_{pq} + g_{pq}\Box - \nabla_p\nabla_q\right]\partial_{\mathcal{R}}F(\mathcal{R}) \;-\; \frac{1}{2}g_{pq}\mathcal{F}(\mathcal{R}) \;=\; +T_{pq} \tag{104}$$

or, equivalently,

$$\left[\nabla^p\nabla^q - g^{pq}\Box - \mathcal{R}^{pq}\right]\partial_{\mathcal{R}}F(\mathcal{R}) \;+\; \frac{1}{2}g^{pq}\mathcal{F}(\mathcal{R}) \;=\; -T^{pq}\;. \tag{105}$$

One can verify that this is in perfect agreement with Eq. (99), given that

$$\begin{aligned}\mathcal{X}^{apqb} &= \frac{\partial F(\mathcal{R})}{\partial \mathcal{R}_{apqb}} \\ &= \frac{1}{2}\left[g^{aq}g^{pb} - g^{ab}g^{pq}\right]\partial_{\mathcal{R}}F(\mathcal{R})\;. \end{aligned}\tag{106}$$

# 6 Example calculations

## 6.1 General considerations

We wish to demonstrate by specific examples that the coefficients of the $h_{rt}$ kinetic terms can indeed be used to directly extract the Wald entropy. We will use two gravitational theories,

$$\mathcal{L}_\alpha \;=\; \mathcal{R} \;+\; \alpha\mathcal{R}^{abcd}\mathcal{R}_{abcd}\;, \tag{107}$$

$$\mathcal{L}_\beta \;=\; \mathcal{R} \;+\; \beta\nabla^k R^{abcd}\nabla_k R_{abcd}\;, \tag{108}$$

where $\alpha$ and $\beta$ are constants. From a physically motivated perspective, these parameters should be regarded as small: $\alpha \ll r_h^2$ and $\beta \ll r_h^4$ , where $r_h$ is the horizon radius of the corresponding black hole solution.

The Einstein term $\mathcal{R}$ allows us to normalize our results and, more importantly, given that the corrections to Einstein gravity are small, it allows us to use the Einstein background solution in the calculations. Any contribution from a perturbative correction is already first

order in $\alpha$ or $\beta$, so that the Einstein background solution suffices. The kinetic coefficients in Einstein's theory are numerical constants and so are insensitive to the form of the solution. [11]

In the following, we incorporate the notation $\overline{g}_{ab} = g^{(0)}_{ab}$ , $\overline{\mathcal{R}}_{abcd} = \mathcal{R}^{(0)}_{abcd}$ and $\overline{\Gamma}_{ab,c} = \Gamma^{(0)}_{ab,c}$ , where $\Gamma_{ab,c} = g_{cd}\Gamma^{d}_{ab}$ .

## 6.2 4-derivative gravity

Let us first consider the theory Eq. (107). Keeping in mind that the Lagrangian is integrated to get an action, $I = \int \sqrt{-g}\mathcal{L}_\alpha d^D x$, we can integrate by parts.

To begin, we write the linear in $\alpha$ and quadratic in $h$ part of the Lagrangian density

$$\delta\left[\sqrt{g}\mathcal{L}_\alpha\right]^{(2)} \;=\; \alpha\delta\left[\sqrt{-g}g^{a\alpha}g^{b\beta}g^{c\gamma}g^{d\delta}\mathcal{R}_{\alpha\beta\gamma\delta}\mathcal{R}_{abcd}\right]^{(2)} \tag{109}$$

and look for any term that makes a kinetic contribution. As a first step, let us consider the "(1,1)" terms or $\delta\left[\sqrt{-g}g^{a\alpha}g^{b\beta}g^{c\gamma}g^{d\delta}\mathcal{R}_{\alpha\beta\gamma\delta}\right]^{(1)}\delta\mathcal{R}^{(1)}_{abcd}$. Our previous analysis suggests that it does not contribute. We will demonstrate this explicitly by working with the unexpanded version of the factor $\sqrt{-g}g^{a\alpha}g^{b\beta}g^{c\gamma}g^{d\delta}\mathcal{R}_{\alpha\beta\gamma\delta}$. So we rewrite Eq. (109) as

$$\delta\left[\sqrt{g}\mathcal{L}_\alpha\right]^{(2)} \;==\alpha\sqrt{-g}g^{a\alpha}g^{b\beta}g^{c\gamma}g^{d\delta}\mathcal{R}_{\alpha\beta\gamma\delta}\left[\overline{\nabla}_b\overline{\nabla}_c h_{ad} - \overline{\nabla}_b\overline{\nabla}_d h_{ac} - \overline{\nabla}_a\overline{\nabla}_c h_{bd} + \overline{\nabla}_a\overline{\nabla}_d h_{bc}\right] \;, \tag{110}$$

where Eq. (51) has been used to expand out $\delta\mathcal{R}^{(1)}_{abcd}$.

Now, to order $h$ in the square brackets, any of the $\overline{\nabla}$'s can be replaced with a "full" $\nabla$. Let us do so with the left-most $\overline{\nabla}$ in each of the four terms and then integrate by parts:

$$-\alpha\sqrt{-g}g^{a\alpha}g^{b\beta}g^{c\gamma}g^{d\delta}\left[\nabla_b\mathcal{R}_{\alpha\beta\gamma\delta}\overline{\nabla}_c h_{ad} - \nabla_b\mathcal{R}_{\alpha\beta\gamma\delta}\overline{\nabla}_d h_{ac} - \nabla_a\mathcal{R}_{\alpha\beta\gamma\delta}\overline{\nabla}_c h_{bd} + \nabla_a\mathcal{R}_{\alpha\beta\gamma\delta}\overline{\nabla}_d h_{bc}\right] \;. \tag{111}$$

As in Section 2, when two $\nabla$'s act on a graviton, the Killing relation can be used to convert them to a Riemann tensor. So, to get a kinetic term out of Eq. (111), we need to

[11] Here, it is assumed that the intention is to calculate the Wald entropy in units of horizon area. The Einstein contribution to the horizon area does get corrected.

expand $\mathcal{R}_{\alpha\beta\gamma\delta}$ to first order in $h$ and to zeroth order in derivatives. Then, since $\mathcal{R}_{abcd} = \partial_b\Gamma_{ac,d} - \partial_a\Gamma_{bc,d} + g^{mn}\Gamma_{ac,m}\Gamma_{nb,d} - g^{mn}\Gamma_{bc,m}\Gamma_{na,d}$ and $\Gamma_{ab,c} = \frac{1}{2}\left(\partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab}\right)$ , the only terms in $\mathcal{R}_{abcd}$ that can possibly contribute are $g^{mn}\overline{\Gamma}_{ac,m}\overline{\Gamma}_{nb,d}$ and $g^{mn}\overline{\Gamma}_{bc,m}\overline{\Gamma}_{na,d}$ . But $\nabla_b g^{mn} = \nabla_a g^{mn} = 0$ , so there can be no kinetic contribution at all.

Hence, we have

$$\delta\left[\sqrt{g}\mathcal{L}_\alpha\right]^{(2)} \;=\; 2\alpha\left[\sqrt{-g}\mathcal{R}^{abcd}\right]^{(0)}\delta\mathcal{R}^{(2)}_{abcd} + \cdots \;, \tag{112}$$

or, using Eq. (64),

$$\delta\left[\sqrt{g}\mathcal{L}_\alpha\right]^{(2)} \;=\; 2\alpha\sqrt{-\overline{g}}\overline{\mathcal{R}}^{abcd}\left[\overline{\nabla}_m h_{bc}\overline{\nabla}^m h_{ad} + 2\overline{\nabla}^m h_{ac}\overline{\nabla}_b h_{dm}\right] \;, \tag{113}$$

where the second term in the square brackets can be gauged away.

We can identify $s_W \;=\; 2\alpha\mathcal{C}\overline{\mathcal{R}}^{abcd}\epsilon_{ab}\epsilon_{cd}$ , with the normalization $\mathcal{C} = -2\pi$ given by the Einstein term. This is in agreement with that obtained via a direct application of Wald's formula.

### 6.3 6-derivative gravity

The Lagrangian is now given by Eq. (108). Here, we begin the quadratic density

$$\delta\left[\sqrt{g}\mathcal{L}_\alpha\right]^{(2)} \;=\; \beta\delta\left[\sqrt{-g}g^{a\alpha}g^{b\beta}g^{c\gamma}g^{d\delta}\nabla_k\mathcal{R}_{\alpha\beta\gamma\delta}\nabla^k\mathcal{R}_{abcd}\right]^{(2)} \;. \tag{114}$$

The (1,1) terms, again, cannot contribute. Because of the derivatives in front of the Riemann tensors, either of these must be expanded to first order in $h$ and zeroth order in derivatives. Hence, by the very same reasoning as provided above, a kinetic contribution can not be obtained.

This leaves us, after integration by parts, with

$$\delta\left[\sqrt{g}\mathcal{L}_\beta\right]^{(2)} \;=\; -2\beta\left[\sqrt{-g}\nabla^k\nabla_k\mathcal{R}^{abcd}\right]^{(0)}\delta\mathcal{R}^{(2)}_{abcd} + \cdots \;, \tag{115}$$

or, like before,

$$\delta\left[\sqrt{g}\mathcal{L}_\alpha\right]^{(2)} \;=\; -2\beta\sqrt{-\overline{g}}\overline{\nabla}^k\overline{\nabla}_k\overline{\mathcal{R}}^{abcd}\left[\overline{\nabla}_m h_{bc}\overline{\nabla}^m h_{ad} + 2\overline{\nabla}^m h_{ac}\overline{\nabla}_b h_{dm}\right] + \cdots \;, \tag{116}$$

and so the identification $s_W = -2\beta\mathcal{C}\Box\overline{\mathcal{R}}^{abcd}\epsilon_{ab}\epsilon_{cd}$ follows. This is, once again, in agreement with the Wald formula when the normalization is $\mathcal{C} = -2\pi$ .

Finally, more derivatives beyond six would neither conceptually nor technically complicate the calculation.

# 7 Conclusion

To summarize, we have investigated Wald's Noether charge entropy [3, 4], relying on its identification with a quarter of the horizon area in units of the effective gravitational coupling, as first established in [5]. The Wald entropy can, as now verified, be determined from the kinetic coefficients for the $h_{rt}$ gravitons on the horizon. We have also clarified what terms in the quadratically expanded action (or linearized field equation) can contribute to the entropy and illustrated our general procedure with some of explicit examples. Additionally, we have reconsidered the gravitational field equation for a general theory of gravity and presented it in a form which differs from [4].

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